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ON VIBRATIONS OF RECTANGULAR PARALLELEPIPEDS

R. D. MINDLIN
P.O. Box 385, Grantham, New Hampshire, U.S.A.

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Abstract—An exact, closed-form solution of the three-dimensional equations of elasticity is given for vibrations of rectangular parallelepipeds with traction-free faces.

LAMÉ POTENTIAL FUNCTIONS

The problem of finding exact, closed-form solutions of the three-dimensional equations of linear elasticity, for steady vibrations of homogeneous, isotropic, rectangular parallelepipeds with all faces free of traction, was introduced by Lamé in 1852[1]. He exhibited a solution of that type for a family of modes with locally equivoluminal deformation but, for the case of coupled equivoluminal and dilatational deformations, he proposed a solution in which normal components of traction remain on all six faces. Some progress was made toward a solution with all faces traction free with the discovery, in 1960[2], of an exact, closed-form solution for a family of coupled modes in bars of rectangular cross-section with the four side-faces of the bars traction free. In this paper, that solution is extended to accommodate a family of modes in rectangular parallelepipeds with all six faces free of traction. Whereas the solution for the bars could be expressed in terms of three Lamé potential functions, 18 are required for the parallelepipeds, as follows:

$$\begin{aligned}
\varphi &= A_1 \sin \xi_2 x \sin \eta_3 y \sin \zeta_3 z + A_2 \sin \xi_3 x \sin \eta_2 y \sin \zeta_3 z \\
&\quad + A_3 \sin \xi_3 x \sin \eta_3 y \sin \zeta_2 z \\
H_x &= B_2 \sin \xi_3 x \cos \eta_1 y \cos \zeta_3 z + C_3 \sin \xi_3 x \cos \eta_3 y \cos \zeta_1 z \\
&\quad + D_1 \sin \xi_1 x \cos \eta_3 y \cos \zeta_3 z + E_1 \sin \xi_2 x \cos \eta_3 y \cos \zeta_2 z \\
&\quad + F_1 \sin \xi_2 x \cos \eta_2 y \cos \zeta_3 z \\
H_y &= B_3 \cos \xi_3 x \sin \eta_3 y \cos \zeta_1 z + C_1 \cos \xi_1 x \sin \eta_3 y \cos \zeta_3 z \\
&\quad + D_2 \cos \xi_3 x \sin \eta_1 y \cos \zeta_3 z + E_2 \cos \xi_2 x \sin \eta_2 y \cos \zeta_3 z \\
&\quad + F_2 \cos \xi_3 x \sin \eta_2 y \cos \zeta_2 z \\
H_z &= B_1 \cos \xi_1 x \cos \eta_3 y \sin \zeta_3 z + C_2 \cos \xi_3 x \cos \eta_1 y \sin \zeta_3 z \\
&\quad + D_3 \cos \xi_3 x \cos \eta_3 y \sin \zeta_1 z + E_3 \cos \xi_3 x \cos \eta_2 y \sin \zeta_2 z \\
&\quad + F_3 \cos \xi_2 x \cos \eta_3 y \sin \zeta_2 z
\end{aligned} \tag{1}$$

in which a time-dependent factor $e^{i\omega t}$ is omitted.

The Lamé functions are governed by the equations

$$\begin{aligned}
v_1^2 \nabla^2 \varphi + \omega^2 \varphi &= 0 \\
v_2^2 \nabla^2 (H_x, H_y, H_z) + \omega^2 (H_x, H_y, H_z) &= 0
\end{aligned} \tag{2}$$

where v_1 and v_2 are the velocities of dilatational and equivoluminal waves, respectively, in an infinite medium

$$v_1^2 = (\lambda + 2\mu)/\rho, \qquad v_2^2 = \mu/\rho \tag{3}$$

and λ and μ are Lamé's constants of elasticity, related to Poisson's ratio, ν, in a form to be employed subsequently

$$1 - \mu/(\lambda + 2\mu) = 1/2(1-\nu) = \kappa^2. \tag{4}$$

Upon substituting eqn (1) in eqn (2) we find the requirements

$$\begin{aligned}
&\xi_2^2 + \eta_3^2 + \zeta_3^2 = \xi_3^2 + \eta_2^2 + \zeta_3^2 = \xi_3^2 + \eta_3^2 + \zeta_2^2 = \omega^2/v_1^2 \\
&\xi_3^2 + \eta_1^2 + \zeta_3^2 = \xi_3^2 + \eta_3^2 + \zeta_1^2 = \xi_1^2 + \eta_3^2 + \zeta_3^2 \\
&= \xi_2^2 + \eta_3^2 + \zeta_2^2 = \xi_2^2 + \eta_2^2 + \zeta_3^2 = \xi_3^2 + \eta_2^2 + \zeta_2^2 = \omega^2/v_2^2.
\end{aligned}$$

DISPLACEMENTS AND STRESSES

The components of displacement are related to the Lamé functions according to

$$\begin{aligned}
u &= \partial\varphi/\partial x + \partial H_z/\partial y - \partial H_y/\partial z \\
v &= \partial\varphi/\partial y + \partial H_x/\partial z - \partial H_z/\partial x \\
w &= \partial\varphi/\partial z + \partial H_y/\partial x - \partial H_x/\partial y
\end{aligned} \tag{6}$$

whence

$$\begin{aligned}
u = {} & A_1\xi_2 \cos\xi_2 x \sin\eta_3 y \sin\zeta_3 z + (C_1\zeta_3 - B_1\eta_3)\cos\xi_1 x \sin\eta_3 y \sin\zeta_3 z \\
& + A_2\xi_3 \cos\xi_3 x \sin\eta_2 y \sin\zeta_3 z + (D_2\zeta_3 - C_2\eta_1)\cos\xi_3 x \sin\eta_1 y \sin\zeta_3 z \\
& + A_3\xi_3 \cos\xi_3 x \sin\eta_3 y \sin\zeta_2 z + (B_3\zeta_1 - D_3\eta_3)\cos\xi_3 x \sin\eta_3 y \sin\zeta_1 z \\
& + E_2\zeta_3 \cos\xi_2 x \sin\eta_2 y \sin\zeta_3 z + F_2\zeta_2 \cos\xi_3 x \sin\eta_2 y \sin\zeta_2 z \\
& - E_3\eta_2 \cos\xi_3 x \sin\eta_2 y \sin\zeta_2 z - F_3\eta_3 \cos\xi_2 x \sin\eta_3 y \sin\zeta_2 z \\[1ex]
v = {} & A_2\eta_2 \sin\xi_3 x \cos\eta_2 y \sin\zeta_3 z + (C_2\xi_3 - B_2\zeta_3)\sin\xi_3 x \cos\eta_1 y \sin\zeta_3 z \\
& + A_3\eta_3 \sin\xi_3 x \cos\eta_3 y \sin\zeta_3 z + (D_3\xi_3 - C_3\zeta_1)\sin\xi_3 x \cos\eta_3 y \sin\zeta_1 z \\
& + A_1\eta_3 \sin\xi_2 x \cos\eta_3 y \sin\zeta_3 z + (B_1\xi_1 - D_1\zeta_3)\sin\xi_1 x \cos\eta_3 y \sin\zeta_3 z \\
& + E_3\xi_3 \sin\xi_3 x \cos\eta_2 y \sin\zeta_2 z + F_3\xi_2 \sin\xi_2 x \cos\eta_3 y \sin\zeta_2 z \\
& - E_1\zeta_2 \sin\xi_2 x \cos\eta_3 y \sin\zeta_2 z - F_1\zeta_3 \sin\xi_2 x \cos\eta_2 y \sin\zeta_3 z \\[1ex]
w = {} & A_3\zeta_2 \sin\xi_3 x \sin\eta_3 y \cos\zeta_2 z + (C_3\eta_3 - B_3\xi_3)\sin\xi_3 x \sin\eta_3 y \cos\zeta_1 z \\
& + A_1\zeta_3 \sin\xi_2 x \sin\eta_3 y \cos\zeta_3 z + (D_1\eta_3 - C_1\xi_1)\sin\xi_1 x \sin\eta_3 y \cos\zeta_3 z \\
& + A_2\zeta_3 \sin\xi_3 x \sin\eta_2 y \cos\zeta_3 z + (B_2\eta_1 - D_2\xi_3)\sin\xi_3 x \sin\eta_1 y \cos\zeta_3 z \\
& + E_1\eta_3 \sin\xi_2 x \sin\eta_3 y \cos\zeta_2 z + F_1\eta_2 \sin\xi_2 x \sin\eta_2 y \cos\zeta_3 z \\
& - E_2\xi_2 \sin\xi_2 x \sin\eta_2 y \cos\zeta_3 z - F_2\xi_3 \sin\xi_3 x \sin\eta_2 y \cos\zeta_2 z.
\end{aligned} \tag{7}$$

The components of stress are calculated from the displacements by

$$\begin{aligned}
\sigma_{xx} &= \lambda\Delta + 2\mu\partial u/\partial x, \qquad \sigma_{yz} = \mu(\partial w/\partial y + \partial v/\partial z) \\
\sigma_{yy} &= \lambda\Delta + 2\mu\partial v/\partial y, \qquad \sigma_{xz} = \mu(\partial u/\partial z + \partial w/\partial x)
\end{aligned}$$

$$\sigma_{zz} = \lambda\Delta + 2\mu\partial w/\partial z, \qquad \sigma_{xy} = \mu(\partial v/\partial x + \partial u/\partial y) \tag{8}$$

where $\Delta = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z$.

Then, the normal components of stress are given by

$$\begin{aligned}
\mu^{-1}\sigma_{xx} = &-[A_1(\xi_1^2 - \eta_3^2 - \zeta_3^2)\sin\xi_2 x + 2(C_1\zeta_3 - B_1\eta_3)\xi_1\sin\xi_1 x]\sin\eta_3 y\sin\zeta_3 z \\
&-[A_2(\eta_1^2 - 2\eta_2^2 + \xi_3^2 - \zeta_3^2)\sin\xi_3 x + 2E_2\zeta_3\xi_2\sin\xi_2 x]\sin\eta_2 y\sin\zeta_3 z \\
&-[A_3(\zeta_1^2 - 2\zeta_2^2 - \eta_3^2 + \xi_3^2)\sin\xi_3 x - 2F_3\eta_3\xi_2\sin\xi_2 x]\sin\eta_3 y\sin\zeta_2 z \\
&-2[D_2\zeta_3 - C_2\eta_1]\xi_3\sin\xi_3 x\sin\eta_1 y\sin\zeta_3 z \\
&-2[B_3\zeta_1 - D_3\eta_3]\xi_3\sin\xi_3 x\sin\eta_3 y\sin\zeta_1 z \\
&-2[F_2\zeta_2 - E_3\eta_2]\xi_3\sin\xi_3 x\sin\eta_2 y\sin\zeta_2 z
\end{aligned}$$

$$\begin{aligned}
\mu^{-1}\sigma_{yy} = &-[A_1(\xi_1^2 - 2\xi_2^2 - \zeta_3^2 + \eta_3^2)\sin\eta_3 y - 2F_1\zeta_3\eta_2\sin\eta_2 y]\sin\xi_2 x\sin\zeta_3 z \\
&-[A_2(\eta_1^2 - \zeta_3^2 - \xi_3^2)\sin\eta_2 y + 2(C_2\xi_3 - B_2\zeta_3)\eta_1\sin\eta_1 y]\sin\xi_3 x\sin\zeta_3 z \\
&-[A_3(\zeta_1^2 - 2\zeta_2^2 + \eta_3^2 - \xi_3^2)\sin\eta_3 y + 2E_3\xi_3\eta_2\sin\eta_2 y]\sin\xi_3 x\sin\zeta_2 z \\
&-2[D_3\xi_3 - C_3\zeta_1]\eta_3\sin\xi_3 x\sin\eta_3 y\sin\zeta_1 z \\
&-2[B_1\xi_1 - D_1\zeta_3]\eta_3\sin\xi_1 x\sin\eta_3 y\sin\zeta_3 z \\
&-2[F_3\xi_2 - E_1\zeta_2]\eta_3\sin\xi_2 x\sin\eta_3 y\sin\zeta_2 z
\end{aligned}$$

$$\begin{aligned}
\mu^{-1}\sigma_{zz} = &-[A_1(\xi_1^2 - 2\xi_2^2 + \zeta_3^2 - \eta_3^2)\sin\zeta_3 z + 2E_1\eta_3\zeta_2\sin\zeta_2 z]\sin\xi_2 x\sin\eta_3 y \\
&-[A_2(\eta_1^2 - 2\eta_2^2 - \xi_3^2 + \zeta_3^2)\sin\zeta_3 z - 2F_2\xi_3\zeta_2\sin\zeta_2 z]\sin\xi_3 x\sin\eta_2 y \\
&-[A_3(\zeta_1^2 - \eta_3^2 - \xi_3^2)\sin\zeta_2 z + 2(C_3\eta_3 - B_3\xi_3)\zeta_1\sin\zeta_1 z]\sin\xi_3 x\sin\eta_3 y \\
&-2[D_1\eta_3 - C_1\xi_1]\zeta_3\sin\xi_1 x\sin\eta_3 y\sin\zeta_3 z \\
&-2[B_2\eta_1 - D_2\xi_3]\zeta_3\sin\xi_3 x\sin\eta_1 y\sin\zeta_3 z \\
&-2[F_1\eta_2 - E_2\xi_2]\zeta_3\sin\xi_2 x\sin\eta_2 y\sin\zeta_3 z
\end{aligned} \tag{9}$$

and the tangential components of stress are given by

$$\begin{aligned}
\mu^{-1}\sigma_{yz} = &\{2A_1\zeta_3\eta_3\sin\xi_2 x + [D_1(\eta_3^2 - \zeta_3^2) + B_1\xi_1\zeta_3 - C_1\xi_1\eta_3]\sin\xi_1 x\}\cos\eta_3 y\cos\zeta_3 z \\
&+\{2A_2\zeta_3\eta_2\sin\xi_3 x - [E_2\xi_2\eta_2 - F_1(\eta_2^2 - \zeta_3^2)]\sin\xi_2 x\}\cos\eta_2 y\cos\zeta_3 z \\
&+\{2A_3\zeta_2\eta_3\sin\xi_3 x + [F_3\xi_2\zeta_2 + E_1(\eta_3^2 - \zeta_2^2)]\sin\xi_2 x\}\cos\eta_3 y\cos\zeta_2 z \\
&+[B_2(\eta_1^2 - \zeta_3^2) + C_2\xi_3\zeta_3 - D_2\xi_3\eta_1]\sin\xi_3 x\cos\eta_1 y\cos\zeta_3 z \\
&+[C_3(\eta_3^2 - \zeta_1^2) + D_3\xi_3\zeta_1 - B_3\xi_3\eta_3]\sin\xi_3 x\cos\eta_3 y\cos\zeta_1 z \\
&+(E_3\xi_3\zeta_2 - F_2\xi_3\eta_2)\sin\xi_3 x\cos\eta_2 y\cos\zeta_2 z
\end{aligned}$$

$$\begin{aligned}
\mu^{-1}\sigma_{zx} = &\{2A_1\xi_2\zeta_3\sin\eta_3 y + [F_1\eta_2\xi_2 + E_2(\zeta_3^2 - \xi_2^2)]\sin\eta_2 y\}\cos\xi_2 x\cos\zeta_3 z \\
&+\{2A_2\xi_3\zeta_3\sin\eta_2 y + [D_2(\zeta_3^2 - \xi_3^2) + B_2\eta_1\xi_3 - C_2\eta_1\zeta_3]\sin\eta_1 y\}\cos\xi_3 x\cos\zeta_3 z \\
&+\{2A_3\xi_3\zeta_2\sin\eta_3 y - [E_3\eta_2\zeta_2 - F_2(\zeta_2^2 - \xi_3^2)]\sin\eta_2 y\}\cos\xi_3 x\cos\zeta_2 z \\
&+[B_3(\zeta_1^2 - \zeta_3^2) + C_3\eta_3\xi_3 - D_3\eta_3\zeta_1]\cos\xi_3 x\sin\eta_3 y\cos\zeta_1 z \\
&+[C_1(\zeta_3^2 - \xi_1^2) + D_1\eta_3\xi_1 - B_1\eta_3\zeta_3]\cos\xi_1 x\sin\eta_3 y\cos\zeta_3 z \\
&+(E_1\eta_3\xi_2 - F_3\eta_3\zeta_2)\cos\xi_2 x\sin\eta_3 y\cos\zeta_2 z
\end{aligned}$$

$$\begin{aligned}\mu^{-1}\sigma_{xy} = &\{2A_1\xi_2\eta_3 \sin\zeta_3 z - [E_1\zeta_2\xi_2 - F_3(\xi_2^2 - \eta_3^2)\sin\zeta_2 z\}\cos\xi_2 x\cos\eta_3 y \\ &+\{2A_2\xi_3\eta_2\sin\zeta_3 z + [F_2\zeta_2\eta_2 + E_3(\xi_3^2 - \eta_2^2)]\sin\zeta_2 z\}\cos\xi_3 x\cos\eta_2 y \\ &+\{2A_3\xi_3\eta_3\sin\zeta_2 z + [D_3(\xi_3^2 - \eta_3^2) + B_3\zeta_1\eta_3 - C_3\zeta_1\xi_3]\sin\zeta_1 z\}\cos\xi_3 x\cos\eta_3 y \\ &+[B_1(\xi_1^2 - \eta_3^2) + C_1\zeta_3\eta_3 - D_1\zeta_3\xi_1]\cos\xi_1 x\cos\eta_3 y\sin\zeta_3 z \\ &+[C_2(\xi_3^2 - \eta_1^2) + D_2\zeta_3\eta_1 - B_2\zeta_3\xi_3]\cos\xi_3 x\cos\eta_1 y\sin\zeta_3 z \\ &+(E_2\zeta_3\eta_2 - F_1\zeta_3\xi_2)\cos\xi_2 x\cos\eta_2 y\sin\zeta_3 z. \end{aligned} \tag{10}$$

BOUNDARY CONDITIONS

The conditions for the boundaries, $x = \pm a$, $y = \pm b$, $z = \pm c$ of a rectangular parallelepiped, to be traction free are

$$\begin{aligned} \sigma_{xx} = \sigma_{xy} = \sigma_{xz} = 0 \quad &\text{on} \quad x = \pm a \\ \sigma_{yx} = \sigma_{yy} = \sigma_{yz} = 0 \quad &\text{on} \quad y = \pm b \\ \sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0 \quad &\text{on} \quad z = \pm c. \end{aligned} \tag{11}$$

The tangential components of traction vanish on the boundaries if

$$\xi_i = l_i\pi/2a, \qquad \eta_i = m_i\pi/2b, \qquad \zeta_i = n_i\pi/2c \tag{12}$$

for $i = 1, 2, 3$ and l_i, m_i, n_i odd integers. These integers give the integral numbers of half wavelengths of displacement between opposing faces of the body. From eqns (3)–(5) and (12), we find a frequency ratio Ω given by

$$\begin{aligned} \kappa^2\Omega^2 &= l_1^2 - l_2^2 = a^2(m_1^2 - m_2^2)/b^2 = a^2(n_1^2 - n_2^2)/c^2 \\ &= l_2^2 - l_3^2 = a^2(m_2^2 - m_3^2)/b^2 = a^2(n_2^2 - n_3^2)/c^2 \end{aligned} \tag{13}$$

where

$$\Omega^2 = \omega^2/\omega_a^2, \qquad \omega_a^2 = \pi^2 v_2^2/4a^2. \tag{14}$$

We note, from eqn (13), the requirements

$$l_1^2 - l_2^2 = l_2^2 - l_3^2, \qquad m_1^2 - m_2^2 = m_2^2 - m_3^2, \qquad n_1^2 - n_2^2 = n_2^2 - n_3^2 \tag{15}$$

and the results

$$\frac{b^2}{a^2} = \frac{m_1^2 - m_2^2}{l_1^2 - l_2^2} = \frac{m_2^2 - m_3^2}{l_2^2 - l_3^2}, \qquad \frac{c^2}{a^2} = \frac{n_1^2 - n_2^2}{l_1^2 - l_2^2} = \frac{n_2^2 - n_3^2}{l_2^2 - l_3^2}. \tag{16}$$

Although the Lamé functions in eqn (1) produce displacements antisymmetric with respect to all three coordinate planes, symmetry with respect to any or all of the coordinate planes would result from interchange of appropriate sines and cosines. Then the corresponding odd integers, among l_i, m_i, n_i, would be replaced by even integers. Thus, both the frequency, in eqn (13), and the dimensional ratios, in eqn (16), depend on differences between the squares of two integers, say P_i and Q_i, $Q_i > P_i$, both odd or both even. Such differences are positive integral multiples of 4

$$Q_i^2 - P_i^2 = 4N, \qquad N = 1, 2, 3 \ldots \tag{17}$$

It was shown in Ref. [2] that all such P_i and Q_i are given by

$$P_i = N/M_i - M_i, \qquad Q_i = N/M_i + M_i, \qquad M_i = 1, 2, 3 \ldots \leqslant N^{1/2}. \tag{18}$$

These results gave the integers eligible for use in Ref. [2], but here there is the additional restriction, eqn (15). Thus, only those values of N are admissible that give at least two pairs of integers, say Q_i, P_i and Q_j, P_j, which satisfy

$$Q_i^2 - P_i^2 = Q_j^2 - P_j^2, \qquad P_i = Q_j. \tag{19}$$

In the range $0 < N < 1000$, only 34 values of N, as listed in Table 1, give Q and P, from eqn (18), which satisfy eqn (19). One of them ($N = 210$) gives two such combinations and another ($N = 840$) gives three.

For the boundary conditions on the normal components of traction to be satisfied, the six expressions in brackets in each of σ_{xx}, σ_{yy}, σ_{zz}, in eqn (9), must vanish on $x = \pm a$, $y = \pm b$, $z = \pm c$, respectively. These conditions constitute a system of 18 simultaneous, homogeneous equations on the 18 constants $A_1 \ldots F_3$, the solution of which is

$$\frac{A_2}{A_1} = \frac{a_{23}a_{31}e_{13}e_{21}f_{32}f_{13}}{a_{32}a_{13}e_{23}e_{31}f_{12}f_{23}}, \qquad \frac{A_3}{A_1} = -\frac{a_{31}e_{12}f_{13}}{a_{13}e_{31}f_{23}}$$

$$\begin{aligned} B_1/A_1 &= -a_{11}c_{31}d_{21}/\Delta_1, & C_1/A_1 &= -a_{11}b_{21}d_{31}/\Delta_1 \\ B_2/A_2 &= -a_{22}c_{12}d_{32}/\Delta_2, & C_2/A_2 &= -a_{22}b_{32}d_{12}/\Delta_2 \\ B_3/A_3 &= -a_{33}c_{23}d_{13}/\Delta_3, & C_3/A_3 &= -a_{33}b_{13}d_{23}/\Delta_3 \end{aligned}$$

$$\begin{aligned} D_1/A_1 &= a_{11}b_{21}c_{31}/\Delta_1, & \Delta_1 &= b_{11}c_{31}d_{21} + b_{21}c_{11}d_{31} \\ D_2/A_2 &= a_{22}b_{32}c_{12}/\Delta_2, & \Delta_2 &= b_{22}c_{12}d_{32} + b_{33}c_{22}d_{12} \\ D_3/A_3 &= a_{33}b_{13}c_{23}/\Delta_3, & \Delta_3 &= b_{33}c_{23}d_{13} + b_{13}c_{33}d_{23} \end{aligned}$$

$$\begin{aligned} E_1/A_1 &= -a_{31}/e_{31}, & F_1/A_1 &= -a_{21}/f_{21} \\ F_2/A_2 &= -a_{12}/e_{12}, & F_2/A_2 &= -a_{32}/f_{32} \\ E_3/A_3 &= -a_{23}/e_{23}, & F_3/A_3 &= -a_{13}/f_{13} \end{aligned} \tag{20}$$

provided that

$$\hat{l}_3^2(\hat{l}_3^2 - \hat{n}_3^2)(\hat{m}_3^2 - \hat{l}_3^2) + \hat{m}_3^2(\hat{m}_3^2 - \hat{l}_3^2)(\hat{n}_3^2 - \hat{m}_3^2) + \hat{n}_3^2(\hat{l}_3^2 - \hat{n}_3^2)(\hat{n}_3^2 - \hat{m}_3^2) + \hat{l}_3^2\hat{m}_3^2\hat{n}_3^2 = 0 \tag{21}$$

where

$$\hat{l}_i = l_i/a, \qquad \hat{m}_i = m_i/b, \qquad \hat{n}_i = n_i/c \tag{22}$$

and

$$a_{11} = (\hat{l}_1^2 - \hat{m}_3^2 - \hat{n}_3^2)(-1)^{(l_2 - l_3)/2}, \; b_{11} = -2\hat{l}_1\hat{m}_3, \; c_{11} = 2\hat{n}_3\hat{l}_1$$

$$a_{12} = (\hat{m}_1^2 - 2\hat{m}_2^2 + \hat{l}_3^2 - \hat{n}_3^2)(-1)^{(l_3 - l_2)/2}, \; e_{12} = 2\hat{n}_3\hat{l}_2$$

$$a_{13} = (\hat{n}_1^2 - 2\hat{n}_2^2 - \hat{m}_3^2 + \hat{l}_3^2)(-1)^{(l_3-l_2)/2},\ f_{13} = -2\hat{l}_2\hat{m}_3$$

$$d_{12} = \hat{n}_3,\ c_{12} = -\hat{m}_1,\ b_{13} = \hat{n}_1,\ d_{13} = -\hat{m}_3,\ f_{12} = \hat{n}_2,\ e_{13} = -\hat{m}_2$$

$$a_{21} = (\hat{l}_1^2 - 2\hat{l}_2^2 - \hat{n}_3^2 + \hat{m}_3^2)(-1)^{(m_3-m_2)/2},\ f_{21} = -2\hat{m}_2\hat{n}_3$$

$$a_{22} = (\hat{m}_1^2 - \hat{n}_3^2 - \hat{l}_3^2)(-1)^{(m_2-m_1)/2},\ b_{22} = -2\hat{m}_1\hat{n}_3,\ c_{22} = 2\hat{l}_3\hat{m}_1$$

$$a_{23} = (\hat{n}_1^2 - 2\hat{n}_2^2 + \hat{m}_3^2 - \hat{l}_3^2)(-1)^{(m_3-m_2)/2},\ e_{23} = 2\hat{l}_3\hat{m}_2$$

$$d_{23} = \hat{l}_3,\ c_{23} = -\hat{n}_1,\ b_{21} = \hat{l}_1,\ d_{21} = -\hat{n}_3,\ f_{23} = \hat{l}_2,\ e_{21} = -\hat{n}_2$$

$$a_{31} = (\hat{l}_1^2 - 2\hat{l}_2^2 + \hat{n}_3^2 - \hat{m}_3^2)(-1)^{(n_3-n_2)/2},\ e_{31} = 2\hat{m}_3\hat{n}_2$$

$$a_{32} = (\hat{m}_1^2 - 2\hat{m}_2^2 - \hat{l}_3^2 + \hat{n}_3^2)(-1)^{(n_3-n_2)/2},\ f_{32} = -2\hat{n}_3\hat{l}_3$$

$$a_{33} = (\hat{n}_1^2 - \hat{m}_3^2 - \hat{l}_3^2)(-1)^{(n_2-n_1)/2},\ b_{33} = -2\hat{n}_1\hat{l}_3,\ c_{33} = 2\hat{m}_3\hat{n}_1$$

$$a_{31} = \hat{m}_3,\ c_{31} = -\hat{l}_1,\ b_{32} = \hat{m}_1,\ d_{32} = -\hat{l}_3,\ f_{31} = \hat{m}_2,\ e_{32} = -\hat{l}_2. \tag{23}$$

EXAMPLES

It remains only to select differences of squares of integers, from eqns (17) and (18), to produce the frequencies according to eqn (13) and dimensional ratios according to eqn (16)—subject to restrictions (15), as in Table 1, and, finally, eqn (21).

The density of modes and shapes of parallelepipeds, included in the solution, is low in comparison with that in the solution for the bar in Ref. [2]. There, with no restriction on length, the allowable ratio of width to thickness of a bar was the square root of the ratio of any two of the $Q_i^2 - P_i^2$ differences calculated from eqn (18). In the range $0 < N \leqslant 1000$, there are 3551 such differences. But, in the present solution, only 37 combinations of two of them, as listed in Table 1, are allowed as a result of eqn (15). Three such combinations are required for the l_i, m_i, n_i of any solution and the admissibility of such triplets is severely restricted by eqn (21). That equation may be simplified by taking any one of the differences, say $\hat{m}_3^2 - \hat{n}_3^2$, equal to zero. As a result, eqn (21) reduces to

$$b^2/a^2 = 2m_3^2/l_3^2 \tag{24}$$

and, since we have taken $b^2/c^2 = m_3^2/n_3^2$, we find

$$c^2/a^2 = 2n_3^2/l_3^2. \tag{25}$$

Then, with eqn (16), we can write $2(l_2^2/l_3^2 - 1) = m_2^2/m_3^2 - 1 = n_2^2/n_3^2 - 1$. These relations facilitate the selection of l_i, m_i, n_i from Table 1 or its extension to higher N.

A few examples of modes in square and rectangular plates and bars are given below.

Square, thick plate:

$$N_1 = 30,\quad l_1 = 17,\quad l_2 = 13,\quad l_3 = 7$$

$$N_2 = 540,\quad m_1 = 69,\quad m_2 = 51,\quad m_3 = 21$$

$$N_3 = 540,\quad n_1 = 69,\quad n_2 = 51,\quad n_3 = 21$$

$$a^2{:}b^2{:}c^2 = 1{:}18{:}18$$

$$\kappa^2\Omega^2 = 120.$$

Table 1. $4N = Q_i^2 - P_i^2 = Q_j^2 - P_j^2$, $P_i = Q_j$

N	Q_i	P_i	P_j
6	7	5	1
24	14	10	2
30	17	13	7
54	21	15	3
60	23	17	7
84	31	25	17
96	28	20	4
120	34	26	14
150	35	25	5
180	49	41	31
210	47	37	23
210	41	29	1
216	42	30	6
240	46	34	14
270	51	39	21
294	49	35	7
330	71	61	49
336	62	50	34
384	56	40	8
480	68	52	28
486	63	45	9
504	79	65	47
540	69	51	21
546	97	85	71
600	70	50	10
630	73	53	17
720	98	82	62
726	77	55	11
750	85	65	35
756	93	75	51
840	127	113	97
840	94	74	46
840	82	58	2
864	84	60	12
924	89	65	23
960	92	68	28
990	119	101	79

Square bar:

$$N_1 = 750, \quad l_1 = 85, \quad l_2 = 65, \quad l_3 = 35$$
$$N_2 = 60, \quad m_1 = 23, \quad m_2 = 17, \quad m_3 = 7$$
$$N_3 = 60, \quad n_1 = 23, \quad n_2 = 17, \quad n_3 = 7$$
$$a^2:b^2:c^2 = 25:2:2$$
$$\kappa^2\Omega^2(c^2/a^2) = 240.$$

Rectangular bar:

$$N_1 = 30, \quad l_1 = 17, \quad l_2 = 13, \quad l_3 = 7$$
$$N_2 = 60, \quad m_1 = 23, \quad m_2 = 17, \quad m_3 = 7$$
$$N_3 = 540, \quad n_1 = 69, \quad n_2 = 51, \quad n_3 = 21$$
$$a^2:b^2:c^2 = 1:2:18$$
$$\kappa^2\Omega^2 = 120.$$

Rectangular plate:

$$N_1 = 750, \quad l_1 = 85, \quad l_2 = 65, \quad l_3 = 35$$
$$N_2 = 540, \quad m_1 = 69, \quad m_2 = 51, \quad m_3 = 21$$

$$N_3 = 60, \quad n_1 = 23, \quad n_2 = 17, \quad n_3 = 7$$
$$a^2 : b^2 : c^2 = 25 : 18 : 2$$
$$\kappa^2 \Omega^2 (c^2/a^2) = 240.$$

As mentioned earlier, some or all of the sines and cosines in the Lamé functions may be interchanged with the result that some or all of the l_i, m_i, n_i would change from odd to even integers.

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